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On the Rate of Convergence of Bernstein Polynomials of Functions of Bounded Variation

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1. INTRODUCTION

If f is a function defined on $[0, 1]$, then the Bernstein polynomial $B_n(f)$ of f.

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, P_{kn}(x), \qquad P_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.1}$$

converges to $f(x)$ uniformly on $[0, 1]$ if f is cntinuous there [1]. As to the rate of convergence, T. Popoviciu [2] has shown that

$$|B_n(f, x) - f(x)| \leqslant \tfrac{5}{4}\omega_f(n^{-1/2}), \tag{1.2}$$

where ω_f is the modulous of continuity of f in $[0, 1]$. It is known that (1.2) cannot be asymptotically improved. However, 5/4 can be replaced [9] by $(8306 + 837\sqrt{6})/5832$ which is best.

As for discontinuous function, Herzog and Hill, and others ([3], see also [4]), proved that if f is bounded on $[0, 1]$ and x is a point of discontinuity of the first kind, then

$$\lim_{n\to\infty} B_n(f, x) = \tfrac{1}{2}(f(x+) + f(x-)). \tag{1.3}$$

In particular, if f is of bounded variation on $[0, 1]$, then (1.3) holds for every x in $(0, 1)$.

In this note, we shall give an estimate for the rate of convergence of (1.3) for functions of bounded variation in terms of the arithmetic means of the sequence of total variations and prove that our estimate is essentially the best possible at points of continuity. Results of this type for Fourier series of 2π-periodic functions of bounded variation and for Fourier–Legendre series of functions of bounded variation were proved in [5] and [6].

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2. Results

Let f be a function defined on $[0, 1]$. For any fixed $x \in (0, 1)$, define g_x as follows if both $f(x+)$ and $f(x-)$ exist:

$$
\begin{aligned}
g_x(t) &= f(t) - f(x+), && x < t \leqslant 1, \\
&= 0, && t = x, \\
&= f(t) - f(x-), && 0 \leqslant t < x.
\end{aligned}
$$

g_x is continuous at the point $t = x$. With this definition of g_x and a simple algebra (1.1) can be expressed as

$$
\begin{aligned}
&B_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-)) \\
&\quad = B_n(g_x, x) + \tfrac{1}{2}(f(x+) - f(x-)) \left(\sum_{k > nx} P_{kn}(x) - \sum_{k < nx} P_{kn}(x) \right).
\end{aligned}
$$

Furthermore, if we let $\sigma_c(t) = \operatorname{sign}(t - c)$ then

$$
\begin{aligned}
B_n(\sigma_x, x) &= \sum_{k=0}^{n} \sigma_x(k/n)\, P_{kn}(x) \\
&= \sum_{k > nx} P_{kn}(x) - \sum_{k < nx} P_{kn}(x)
\end{aligned}
$$

and so

$$
B_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-)) = B_n(g_x, x) + \tfrac{1}{2}(f(x+) - f(x-))\, B_n(\sigma_x, x). \tag{2.1}
$$

It shows that to estimate $|B_n(f, x) - \frac{1}{2}(f(x+) + f(x-))|$ we only have to evaluate $B_n(g_x, x)$ and $B_n(\sigma_x, x)$.

Our main result may be stated as follows:

THEOREM. *Let f be of bounded variation on $[0, 1]$ and $V_a^b(g_x)$ be the total variation of g_x on $[a, b]$. Then for every $x \in (0, 1)$ and $n \geqslant (3/x(1-x))^8$ we have*

$$
\begin{aligned}
|B_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-))| \leqslant{} & \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}}(g_x) \\
& + \frac{18(x(1-x))^{-5/2}}{n^{1/6}} |f(x+) - f(x-)|.
\end{aligned} \tag{2.2}
$$

The right-hand side of (2.2) converges to zero as $n \to \infty$ since continuity of g_x at x implies that

$$V_{x-\beta}^{x+\alpha}(g_x) \to 0 \qquad (\alpha, \beta \to 0+).$$

If f is of bounded variation on $[0, 1]$ and continuous at x then the inequality (2.2) becomes

$$|B_n(f, x) - f(x)| \leqslant \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f). \tag{2.3}$$

Let us now consider the function $f(t) = |t - x|$ $(0 < x < 1)$ on $[0, 1]$. We have, for any small δ,

$$\begin{aligned} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right| P_{kn}(x) &\leqslant \left(\sum_{|k/n - x| \leqslant \delta} + \sum_{|k/n - x| > \delta}\right) \left|\frac{k}{n} - x\right| P_{kn}(x) \\ &\leqslant \delta + \frac{1}{\delta} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 P_{kn}(x) \qquad (2.4) \\ &\leqslant \delta + \frac{x(1-x)}{n\delta} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right| P_{kn}(x) &\geqslant \sum_{|k/n - x| \leqslant \delta} \left|\frac{k}{n} - x\right| P_{kn}(x) \\ &\geqslant \frac{1}{\delta} \sum_{|k/n - x| \leqslant \delta} \left(\frac{k}{n} - x\right)^2 P_{kn}(x) \\ &\geqslant \frac{x(1-x)}{n\delta} - \frac{1}{\delta} \sum_{|k/n - x| > \delta} \left(\frac{k}{n} - x\right)^2 P_{kn}(x). \end{aligned}$$

Since

$$\begin{aligned} &\sum_{|k/n - x| > \delta} \left(\frac{k}{n} - x\right)^2 P_{kn}(x) \\ &\quad \leqslant \frac{1}{\delta^2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^4 P_{kn}(x) \\ &\quad \leqslant \frac{1}{\delta^2} \left(\frac{3x^2(1-x)^2}{n^2} + \frac{1}{n^3}\left(x(1-x) - 6x^2(1-x)^2\right)\right) \\ &\quad \leqslant \frac{x^2(1-x)^2}{n^2\delta^2} \left(3 + \frac{1}{nx(1-x)}\right), \end{aligned}$$

it follows that

$$\sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) \geqslant \frac{x(1-x)}{n\delta} - \frac{7}{2} \frac{x^2(1-x)^2}{n^2\delta^2}. \tag{2.5}$$

if $n > 2/(x(1-x))$. Choose $\delta = 2(x(1-x)/n)^{1/2}$, we obtain from (2.4) that

$$\sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) \leqslant \frac{5}{2} \frac{(x(1-x))^{1/2}}{n^{1/2}}$$

and from (2.5) that

$$\begin{aligned} \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) &\geqslant \frac{1}{2} \frac{(x(1-x))^{1/2}}{n^{1/2}} - \frac{7}{8} \frac{x^2(1-x)^2}{n^2(x(1-x)/n)^{3/2}} \\ &\geqslant \frac{1}{16} \frac{(x(1-x))^{1/2}}{n^{1/2}}. \end{aligned}$$

Therefore, if $n > 2/(x(1-x))$ then we have

$$\frac{1}{16} \frac{(x(1-x))^{1/2}}{n^{1/2}} \leqslant \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) \leqslant \frac{5}{2} \frac{(x(1-x))^{1/2}}{n^{1/2}}. \tag{2.6}$$

On the other hand, from (2.3), since $V_{x+\beta}^{x+\alpha}(f) = \alpha - \beta$, it follows that

$$\begin{aligned} |B_n(f, x) - f(x)| = \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) &\leqslant \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f) \\ &\leqslant \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \\ &\leqslant \frac{3(x(1-x))^{-1}}{n^{1/2}}. \end{aligned} \tag{2.7}$$

Hence by comparing (2.6) and (2.7) we see that (2.3) cannot be asymptotically improved for functions of bounded variation at points of continuity as we have mentioned before.

A more precise version of (2.6),

$$\lim_{n \to \infty} n^{1/2} \cdot \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| P_{kn}(x) = (2x(1-x))^{1/2}$$

was proved in [7].

3. Proof of the Theorem: Evaluation of $B_n(\sigma_x, x)$

The convergence of the sequence $B_n(\sigma_x, x)$ to zero as $n \to \infty$ follows immediately from the well-known central limit theorem of probability. However, what we are interested in here is finding an estimate for the rate of convergence of this result. To do so, we first decompose $B_n(\sigma_x, x)$ into three parts as follows:

$$B_n(\sigma_x, x) = A_n(x) - B_n(x) + C_n(x) \tag{3.1}$$

with

$$A_n(x) = \sum_{x < k/n \leqslant x + n^{-\alpha}} P_{kn}(x),$$

$$B_n(x) = \sum_{x - n^{-\alpha} \leqslant k/n < x} P_{kn}(x),$$

$$C_n(x) = \left(- \sum_{0 \leqslant k/n < x - n^{-\alpha}} + \sum_{x + n^{-\alpha} < k/n \leqslant 1} \right) P_{kn}(x),$$

where $0 < \alpha < 1$.

The evaluation of $C_n(x)$ is relatively easy. Observe that

$$|C_n(x)| \leqslant \sum_{|k/n - x| > n^{-\alpha}} P_{kn}(x) \leqslant n^{2\alpha} \cdot \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^2 P_{kn}(x).$$

Since $\sum_{k=0}^{n} (k/n - x)^2 P_{kn}(x) = x(1-x)/n$ it follows that

$$|C_n(x)| \leqslant \frac{x(1-x)}{n^{1-2\alpha}} \leqslant \frac{\frac{1}{4}}{n^{1-2\alpha}}. \tag{3.2}$$[1]

To evaluate $A_n(x)$ and $B_n(x)$, we need a convenient asymptotic form for $P_{kn}(x)$'s satisfying the inequality $|k/n - x| \leqslant n^{-\alpha}$. Using Stirling's formula,

$$n! = (2\pi n)^{1/2} n^n e^{-n} H_n,$$
$$H_n = e^{\theta_n/12n}, \qquad 0 < \theta_n < 1,$$

we obtain

$$P_{kn}(x) = \left(\frac{n}{2\pi k(n-k)} \right)^{1/2} W_{kn}(x) H_{kn}.$$

[1] A more precise version can be found in [8, p. 15]. But the role played by $C_n(x)$ in our proof is not essential. (3.2) is good enough for our needs.

where

$$W_{kn}(x)=\frac{n^n}{k^k(n-k)^{n-k}}x^k(1-x)^{n-k},$$

$$H_{kn}(x)=\frac{H_n}{H_k H_{n-k}}.$$

It is easy to see that if $n>(2/(x(1-x))^{1/\alpha}$ and $|k/n-x|\leqslant n^{-\alpha}$ then

$$|H_{kn}-1|\leqslant\frac{2}{3nx(1-x)}$$

and

$$\left|\left(\frac{n}{2\pi k(n-k)}\right)^{1/2}-\left(\frac{1}{2\pi nx(1-x)}\right)^{1/2}\right|\leqslant\frac{4}{3\sqrt{2\pi}\,n^{\alpha+1/2}(x(1-x))^{3/2}}.$$

On the other hand, since $(n/2\pi k(n-k))^{1/2}\cdot W_{kn}(x)=P_{kn}(x)/H_{kn}$ is uniformly bounded by 2, it follows that

$$\begin{aligned}&\left|P_{kn}(x)-\left(\frac{1}{2\pi nx(1-x)}\right)^{1/2}W_{kn}(x)\right|\\&\quad\leqslant\left|\left(\frac{n}{2\pi k(n-k)}\right)^{1/2}W_{kn}(x)(H_{kn}-1)\right|\\&\qquad+W_{kn}(x)\left|\left(\frac{n}{2\pi k(n-k)}\right)^{1/2}-\left(\frac{1}{2\pi nx(1-x)}\right)^{1/2}\right|\\&\quad\leqslant\frac{4}{3nx(1-x)}+W_{kn}(x)\frac{4}{3\sqrt{2\pi}\,n^{\alpha+1/2}(x(1-x))^{3/2}}\end{aligned}\tag{3.3}$$

if $n>(2/(x(1-x)))^{1/\alpha}$ and $|k/n-x|\leqslant n^{-\alpha}$.

The following lemma, which gives us a precise estimation of $W_{kn}(x)$, is the key to the evaluation of $B_n(\sigma_x, x)$.

LEMMA (LAPLACE'S FORMULA OF PROBABILITY). *If $\frac{1}{3}<\alpha<1$ and $n\geqslant(3/(x(1-x)))^{2/(3\alpha-1)}$ then*

$$\left|W_{kn}(x)-\exp\left[-(2x(1-x))^{-1}n\left(\frac{k}{n}-x\right)^2\right]\right|\leqslant\frac{9}{n^{3\alpha-1}(x(1-x))^2}$$

holds uniformly for all k satifying the inequality $|k/n-x|\leqslant n^{-\alpha}$. In particular, $W_{kn}(x)$ is then bounded by 2.

Proof of the Lemma. By Taylor's formula for $|u| < 1$,

$$\begin{aligned}\log(1+u) &= u - \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3(1+tu)^{-3} \\ &= u - \tfrac{1}{2}u^2[1 - \tfrac{2}{3}u(1+tu)^{-3}] \\ &= u - \tfrac{1}{2}u^2\rho,\end{aligned}$$

$0 < t < 1$, $\rho = 1 - \frac{2}{3}u(1+tu)^{-3} = 1 + \varepsilon u$, where $\varepsilon = -\frac{2}{3}(1+tu)^{-3}$. If $|u| \leqslant \frac{1}{2}$ then $|\varepsilon| \leqslant 16/3$ and $|\rho| \leqslant 11/3$. Similarly we can express $\log(1-u)$ as

$$\log(1-u) = -u - \tfrac{1}{2}u^2\rho_1$$

with $\rho_1 = 1 + \varepsilon_1 u$ for some ε_1 such that $|\varepsilon_1| \leqslant 16/3$ and $|\rho_1| \leqslant 11/3$ if $|u| \leqslant \frac{1}{2}$.

Since

$$-\log W_{kn}(x) = k\log(1 + x^{-1}(k/n - x)) + (n-k)\log(1 - (1-x)^{-1}(k/n - x))$$

and $|x^{-1}(k/n - x)| \leqslant \frac{1}{2}$, $|(1-x)^{-1}(k/n - x)| \leqslant \frac{1}{2}$ if

$$n \geqslant \left(\frac{3}{x(1-x)}\right)^{2/(3\alpha-1)}$$

Therefore

$$\begin{aligned}&-\log W_{kn}(x) \\ &\quad = k(x^{-1}(k/n - x) - \tfrac{1}{2}x^{-2}(k/n - x)^2\rho) \\ &\qquad - (n-k)((1-x)^{-1}(k/n - x) + \tfrac{1}{2}(1-x)^{-2}(k/n - x)^2\rho_1) \\ &\quad = (nx + n(k/n - x))(x^{-1}(k/n - x) - \tfrac{1}{2}x^{-2}(k/n - x)^2\rho) \\ &\qquad - (n(1-x) - n(k/n - x))((1-x)^{-1}(k/n - x) + \tfrac{1}{2}(1-x)^{-2}(k/n - x)^2\rho_1 \\ &\quad = n(k/n - x)^2(x^{-1}(1 - \tfrac{1}{2}\rho - \tfrac{1}{2}x^{-1}\rho(k/n - x)) \\ &\qquad + (1-x)^{-1}(1 - \tfrac{1}{2}\rho_1 + \tfrac{1}{2}(1-x)^{-1}(k/n - x)\rho_1)) \\ &\quad = (2x(1-x))^{-1}n(k/n - x)^2 + n(k/n - x)^2(x^{-1}(\tfrac{1}{2} - \tfrac{1}{2}\rho - \tfrac{1}{2}x^{-1}\rho(k/n - x)) \\ &\qquad + (1-x)^{-1}(\tfrac{1}{2} - \tfrac{1}{2}\rho_1 + \tfrac{1}{2}(1-x)^{-1}\rho_1(k/n - x))) \\ &\quad = (2x(1-x))^{-1}n(k/n - x)^2 + \tfrac{1}{2}n(k/n - x)^3(-x^{-2}(\varepsilon + \rho) \\ &\qquad + (1-x)^{-2}(-\varepsilon_1 + \rho_1))\end{aligned}$$

and so

$$\left|\log W_{kn}(x) + (2x(1-x))^{-1}n\left(\frac{k}{n} - x\right)^2\right| \leqslant \frac{9}{2n^{3\alpha-1}(x(1-x))^2}.$$

But if $\frac{1}{3} < \alpha < 1$ and $n \geqslant (3/x(1-x))^{2/(3\alpha-1)}$ then we have

$$\left|\exp\left(\frac{9}{2n^{3\alpha-1}(x(1-x))^2}\right) - 1\right| \leqslant \frac{9}{n^{3\alpha-1}(x(1-x))^2}.$$

Hence

$$\begin{aligned}
&\left|W_{kn}(x) - \exp\left(-(2x(1-x))^{-1} n \left(\frac{k}{n} - x\right)^2\right)\right| \\
&\qquad \leqslant \exp\left(-(2x(1-x))^{-1} n \left(\frac{k}{n} - x\right)^2\right) \left| \exp\left(\log W_{kn}(x)\right.\right. \\
&\qquad\quad \left.\left. + (2x(1-x))^{-1} n \left(\frac{k}{n} - x\right)^2\right) - 1 \right| \\
&\qquad \leqslant \frac{9}{n^{3\alpha-1}(x(1-x))^2}.
\end{aligned}$$

The boundedness of $W_{kn}(x)$ follows from the fact that

$$\frac{9}{n^{3\alpha-1}(x(1-x))^2} \leqslant 1$$

if $n \geqslant (3/x(1-x))^{2/(3\alpha-1)}$. This completes the proof of the lemma.

Consequently, if $\frac{1}{3} < \alpha < 1$ and $n \geqslant (3/x(1-x))^{2/(3\alpha-1)}$, by Laplace's formula and (3.3) we obtain

$$\begin{aligned}
&\left|P_{kn}(x) - (2\pi n x(1-x))^{-1/2} \exp\left(-(2x(1-x))^{-1} n \left(\frac{k}{n} - x\right)^2\right)\right| \\
&\quad \leqslant |P_{kn}(x) - (2\pi n x(1-x))^{-1/2} W_{kn}(x)| + \left|(2\pi n x(1-x))^{-1/2} W_{kn}(x)\right. \\
&\qquad \left. - (2\pi n x(1-x))^{-1/2} \exp\left(-(2x(1-x))^{-1} n \left(\frac{k}{n} - x\right)^2\right)\right| \\
&\quad \leqslant \frac{4}{3nx(1-x)} + \frac{8}{3\sqrt{2\pi}\, n^{\alpha+1/2}(x(1-x))^{3/2}} + \frac{9}{\sqrt{2\pi}\, n^{3\alpha-1/2}(x(1-x))^{5/2}}
\end{aligned} \tag{3.4}$$

for all k satisfying the inequality $|k/n - x| \leqslant n^{-\alpha}$.

However, to estimate the sums of $P_{kn}(x)$ we need a more convenient form.

With simple algebra we can show that

$$(2\pi nx(1-x))^{-1/2} \exp\left(-(2x(1-x))^{-1} n \left(\frac{k}{n}-x\right)^2\right)$$

$$= \left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \int_{k/n}^{(k+1)/n} \exp\left(-\frac{n}{2x(1-x)}(u-x)^2\right) du$$

$$+ \left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \int_{k/n}^{(k+1)/n} \exp\left(-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right)$$

$$\cdot \left(1-\exp\left(-\frac{n}{2x(1-x)}\left(u-\frac{k}{n}\right)\left(u+\frac{k}{n}-2x\right)\right)\right) du.$$

If $n \geqslant (3/(x(1-x)))^{2/(3\alpha-1)}$ then absolute value of the second term on the right-hand side of the last equation $\leqslant$

$$\leqslant (2\pi nx(1-x))^{-1/2}$$

$$\cdot \max_{k/n \leqslant u \leqslant (k+1)/n} \left|1-\exp\left(-\frac{n}{2x(1-x)}\left(u-\frac{k}{n}\right)\left(u+\frac{k}{n}-2x\right)\right)\right|$$

$$\leqslant (2\pi nx(1-x))^{-1/2} \cdot 2$$

$$\cdot \max_{k/n \leqslant u \leqslant (k+1)/n} \left|-\frac{n}{2x(1-x)}\left(u-\frac{k}{n}\right)\left(u+\frac{k}{n}-2x\right)\right|$$

$$\leqslant \frac{1}{\sqrt{2\pi}\, n^{1/2}(x(1-x))^{3/2}} \cdot \max_{k/n \leqslant u \leqslant (k+1)/n} \left|u+\frac{k}{n}-2x\right|$$

$$\leqslant \frac{3}{\sqrt{2\pi}\, n^{\alpha+1/2}(x(1-x))^{3/2}}.$$

Hence from (3.4) and the above inequality we see that

$$\left|P_{kn}(x) - \left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \int_{k/n}^{(k+1)/n} \exp\left(-\frac{n}{2x(1-x)}(u-x)^2\right) du\right|$$

$$\leqslant \frac{4}{3nx(1-x)} + \frac{17}{3\sqrt{2\pi}\, n^{\alpha+1/2}(x(1-x))^{3/2}} + \frac{9}{\sqrt{2\pi}\, n^{3\alpha-1/2}(x(1-x))^{5/2}}, \tag{3.5}$$

if $n \geqslant (3/x(1-x))^{2/(3\alpha-1)}$ and $|k/n - x| \leqslant n^{-\alpha}(\frac{1}{3} < \alpha < 1)$.

We now apply (3.5) to estimate the sums

$$A_n(x) = \sum_{x < k/n \leqslant x + n^{-\alpha}} P_{kn}(x),$$

assuming that $\frac{3}{8} < \alpha < \frac{1}{2}$.

Let k' and k'' be the smallest and largest of the k resp. which satisfy the inequality $x < k/n \leqslant x + n^{-\alpha}$. Since the number of k's between k' and k'' is at most $[n^{1-\alpha}]$, by (3.5) it follows that

$$\left|A_n(x) - \left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \int_x^{x+n^{-\alpha}} \exp\left(-\frac{n}{2x(1-x)}(u-x)^2\right) du\right|$$

$$\leqslant \left|\left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \left(\int_{x+n^{-\alpha}}^{(k''+1)/n} - \int_x^{k'/n}\right) \exp\left(-\frac{n}{2x(1-x)}(u-x)^2\right) du\right|$$

$$+ \frac{4}{3n^{\alpha}x(1-x)} + \frac{17}{3\sqrt{2\pi}\, n^{2\alpha-1/2}(x(1-x))^{3/2}}$$

$$+ \frac{9}{\sqrt{2\pi}\, n^{4\alpha-3/2}(x(1-x))^{5/2}}$$

$$\leqslant \frac{2}{\sqrt{2\pi}\, n^{1/2}(x(1-x))^{1/2}} + \frac{4}{3n^{\alpha}x(1-x)} + \frac{17}{3\sqrt{2\pi}\, n^{2\alpha-1/2}(x(1-x))^{3/2}}$$

$$+ \frac{9}{\sqrt{2\pi}\, n^{4\alpha-3/2}(x(1-x))^{5/2}}.$$

Moreover, since $(x(1-x))^{-a} \geqslant (x(1-x))^{-b}$ if $a \geqslant b > 0$ and $n^{-1/2} < n^{-\alpha} < n^{-(2\alpha-1/2)} < n^{-(4\alpha-3/2)}$ if $\frac{3}{8} < \alpha < 1$, we find that

$$\left|A_n(x) - \left(\frac{n}{2\pi x(1-x)}\right)^{1/2} \int_x^{x+n^{-\alpha}} \exp\left(-\frac{n}{2x(1-x)}(u-x)^2\right) du\right|$$

$$\leqslant \frac{1}{n^{4\alpha-3/2}(x(1-x))^{5/2}} \left(\frac{2}{\sqrt{2\pi}} + \frac{4}{3} + \frac{17}{3\sqrt{2\pi}} + \frac{9}{\sqrt{2\pi}}\right)$$

$$\leqslant \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}},$$

or

$$\left|A_n(x) - \frac{1}{\sqrt{\pi}} \int_0^{M_n} e^{-v^2}\, dv\right| \leqslant \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}}, \tag{3.6}$$

where $M_n = n^{(1/2)-\alpha}(2x(1-x))^{-1/2}$.

With an easy calculation we can show that

$$\sqrt{\pi}/2\,\sqrt{1-e^{-t^2}} \leqslant \int_0^t e^{-v^2}\, dv, \qquad t \geqslant 0.$$

Therefore

$$1/\sqrt{\pi} \int_t^{\infty} e^{-v^2}\, dv \leqslant \tfrac{1}{2}\left(1 - \sqrt{1-e^{-t^2}}\right).$$

On the other hand, since $1-(1-y)^{1/2} \leqslant y/2$ if $0 \leqslant y < 1$ and $e^{-z} \leqslant (1+z)^{-1}$ if $z \geqslant 0$, it follows that

$$\frac{1}{\sqrt{\pi}} \int_{M_n}^{\infty} e^{-v^2}\, dv \leqslant \frac{1}{4} \frac{1}{1+M_n^2} < \frac{1/8}{n^{1-2\alpha}}.$$

Hence, from (3.6),

$$\begin{aligned}\left| A_n(x) - \frac{1}{2} \right| &\leqslant \frac{1/8}{n^{1-2\alpha}} + \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}} \\ &\leqslant \frac{1/8}{n^{1-2\alpha}(x(1-x))^{5/2}} + \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}}.\end{aligned}$$

However, it is easy to see that, on $(\frac{3}{8}, \frac{1}{2})$, the right-hand side of the last inequality asymptotically drops most rapidly when $\alpha = 5/12$. Therefore, by choosing $\alpha = 5/12$, we get the best estimate for $A_n(x)$, namely,

$$|A_n(x) - \tfrac{1}{2}| \leqslant (65/8)(x(1-x))^{-5/2}/n^{1/6} \tag{3.7}$$

if $n \geqslant (3/x(1-x))^{2/(3\alpha-1)} = (3/x(1-x))^8$.

The evaluation of $B_n(x)$ is similar to that of $A_n(x)$. Repeating the same process we can prove that

$$|B_n(x) - \tfrac{1}{2}| \leqslant (65/8)(x(1-x))^{-5/2}/n^{1/6} \tag{3.8}$$

if $n \geqslant (3/x(1-x))^8$.

Then by (3.1), (3.2), (3.7) and (3.8) with $\alpha = 5/12$ in (3.2) it follows that

$$|B_n(\sigma_x, x)| \leqslant 18(x(1-x))^{-5/2}/n^{1/6}, \qquad n \geqslant (3/x(1-x))^8. \tag{3.9}$$

Evaluation of $B_n(g_x, x)$. As we know, $B_n(g_x, x) = \sum_{k=0}^{n} g_x(k/n)\, P_{kn}(x)$ may be written in the form of a Lebesgue–Stieltjes integral in the variable t

$$\sum_{k=0}^{n} g_x(k/n)\, P_{kn}(x) = \int_0^1 g_x(t)\, d_t\, K_n(x, t) \tag{3.10}$$

with the kernel

$$\begin{aligned} K_n(x, t) &= \sum_{k \leqslant tn} P_{kn}(x), \qquad && 0 < t \leqslant 1, \\ &= 0, && t = 0. \end{aligned}$$

To estimate $\int_0^1 g_x(t)\, d_t K_n(x, t)$, we decompose it into three parts, as follows.

$$\int_0^1 g_x(t)\, d_t K_n(x, t) = L_n(f, x) + M_n(f, x) + R_n(f, x) \tag{3.11}$$

with

$$L_n(f, x) = \int_0^{x - x/\sqrt{n}} g_x(t)\, d_t K_n(x, t),$$

$$M_n(f, x) = \int_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} g_x(t)\, d_t K_n(x, t),$$

$$R_n(f, x) = \int_{x + (1-x)/\sqrt{n}}^{1} g_x(t)\, d_t K_n(x, t).$$

First, we evaluate $M_n(f, x)$. For $t \in [x - x/\sqrt{n},\ x + (1-x)/\sqrt{n}]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x),$$

and so

$$|M_n(f, x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x) \cdot \int_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} d_t K_n(x, t).$$

Since

$$\int_a^b d_t K_n(x, t) \leqslant 1 \qquad \text{for all} \quad [a, b] \subseteq [0, 1],$$

therefore

$$|M_n(f, x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1+x)/\sqrt{n}}(g_x). \tag{3.12}$$

To estimate $L_n(f, x)$, let $y = x - x/\sqrt{n}$ and note that g_x is normalized on $(0, 1)$. Using partial integration for Lebesgue–Stieltjes integral, we find that

$$L_n(f, x) = g_x(y+)\, K_n(x, y+) - \int_0^y \hat{K}_n(x, t)\, d_t\, g_x(t),$$

where $\hat{K}_n(x, t)$ is the normalized form of $K_n(x, t)$. Since

$$K_n(x, y+) = K_n(x, y), \qquad 0 < y \leqslant 1,$$

and

$$|g_x(y+)| = |g_x(y+) - g_x(x)| \leqslant V_{y+}^{x}(g_x),$$

where $V_{y+}^x(g_x) = \lim_{\varepsilon \to 0+} V_{y+\varepsilon}^x(g_x)$, it follows that

$$|L_n(f, x)| \leqslant V_{y+}^x(g_x)\, K_n(x, y) + \int_0^y K_n(x, t)\, d_t(-V_t^x(g_x)).$$

By the well-known inequality

$$K_n(x, t) \leqslant \frac{x(1-x)}{n(x-t)^2}, \qquad 0 \leqslant t < x$$

(see, e.g., [8, p. 6]), and the fact $\hat{K}_n(x, t) \leqslant K_n(x, t)$ on $(0, 1]$, we obtain

$$|L_n(f, x)| \leqslant V_{y+}^x(g_x) \frac{x(1-x)}{n(x-y)^2} + \frac{x(1-x)}{n} \int_{0+}^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)) + \frac{(1-x)^n}{2} V_0^{0+}(g_x).$$

Actually, since $(1-x)^n/2 \leqslant x(1-x)/nx^2$ and

$$\frac{x(1-x)}{n} \int_{0+}^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)) + \frac{x(1-x)}{nx^2} V_0^{0+}(g_x) = \frac{x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)),$$

it follows that

$$|L_n(f, x)| \leqslant V_{y+}^x(g_x) \frac{x(1-x)}{n(x-y)^2} + \frac{x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)).$$

Furthermore, since

$$\int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)) = -\frac{V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \hat{V}_t^x(g_x) \frac{dt}{(x-t)^3},$$

where $\hat{V}_t^x(g_x)$ is the normalized form of $V_t^x(g_x)$ and $\hat{V}_t^x(g_x) = V_t^x(g_x)$, we have

$$|L_n(f, x)| \leqslant \frac{x(1-x)}{n} \left(\frac{V_0^x(g_x)}{x^2} + 2\int_0^{x - x/\sqrt{n}} V_t^x(g_x) \frac{dt}{(x-t)^3} \right).$$

Replacing the variable t in the last integral by $x - x/\sqrt{t}$, we find that

$$\int_0^{x - x/\sqrt{n}} V_t^x(g_x) \frac{dt}{(x-t)^3} = \frac{1}{2x^2} \int_1^n V_{x - x/\sqrt{t}}^x(g_x)\, dt \leqslant \frac{1}{2x^2} \sum_{k=1}^n V_{x - x/\sqrt{k}}^x(g_x).$$

Hence

$$
\begin{aligned}
|L_n(f, x)| &\leqslant \frac{1-x}{nx}\left(V_0^x(g_x) + \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x)\right) \\
&\leqslant \frac{2(1-x)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x) \\
&\leqslant \frac{2}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x).
\end{aligned} \tag{3.13}
$$

To estimate $R_n(f, x)$, let $z = x + (1-x)/\sqrt{n}$ and define $H_n(x, t)$ on $[0, 1]$ as follows:

$$
\begin{aligned}
&H_n(x, t) = 1 - K_n(x, t-), \qquad 0 \leqslant t < 1, \\
&H_n(x, 1) = 0.
\end{aligned}
$$

Then

$$R_n(f, x) = -\int_z^1 g_x(t)\, d_t H_n(x, t).$$

Using partial integration for Lebesgue–Stieltjes integral,

$$R_n(f, x) = g_x(z-)\, H_n(x, z-) + \int_z^1 \hat{H}_n(x, t)\, d_t\, g_x(t),$$

where $\hat{H}_n(x, t)$ is the normalized form of $H_n(x, t)$. Since

$$H_n(x, z-) = H_n(x, z), \qquad 0 \leqslant z < 1,$$

and

$$|g_x(z-)| = |g_x(z-) - g_x(x)| \leqslant V_x^z(g_x),$$

so that

$$|R_n(f, x)| \leqslant V_x^z(g_x)\, H_n(x, z) + \int_z^1 \hat{H}_n(x, t)\, d_t V_x^t(g_x).$$

By inequality

$$H_n(x, t) = \sum_{k \geqslant nt} P_{kn}(x) \leqslant \frac{x(1-x)}{n(x-t)^2}, \qquad x \leqslant t < 1,$$

and the fact that $\hat{H}_n(x,t) \leqslant H_n(x,t)$ on $[0,1)$, we have then

$$|R_n(f,x)| \leqslant V_x^{z-}(g_x)\frac{x(1-x)}{n(x-1)^2} + \frac{x(1-x)}{n}\int_z^1 \frac{1}{(x-t)^2}\,d_t V_x^t(g_x) + \frac{x^n}{2} V_{1-}^1(g_x).$$

But as we did for $L_n(f,x)$, since $x^n/2 \leqslant x(1-x)/n(1-x)^2$ and

$$\frac{x(1-x)}{n}\int_z^{1-} \frac{1}{(x-t)^2}\,d_t V_x^t(g_x) + \frac{x(1-x)}{n(1-x)^2} V_{1-}^1(g_x) = \frac{x(1-x)}{n}\int_z^1 \frac{1}{(x-t)^2}\,d_t(V_x^t(g_x)),$$

We actually have

$$|R_n(f,x)| \leqslant \frac{x(1-x)}{n}\left(\frac{V_x^{z-}(g_x)}{(x-z)^2} + \int_z^1 \frac{1}{(x-t)^2}\,d_t V_x^t(g_x)\right).$$

Using partial integration again

$$\int_z^1 \frac{1}{(x-t)^2}\,d_t V_x^t(g_x) = \frac{V_x^1(g_x)}{(1-x)^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} + 2\int_z^1 \hat{V}_x^t(g_x)\frac{dt}{(t-x)^3},$$

where $\hat{V}_x^t(g_x)$ is the normalized form of $V_x^t(g_x)$ and with the fact that $\hat{V}_x^t(g_x) = V_x^t(g_x)$, the preceding inequality becomes

$$|R_n(f,x)| \leqslant \frac{x(1-x)}{n}\left(\frac{V_x^1(g_x)}{(1-x)^2} + 2\int_z^1 V_x^t(g_x)\frac{dt}{(t-x)^3}\right).$$

Replacing the variable t by $x + (1-x)/\sqrt{t}$,

$$\int_z^1 V_x^t(g_x)\frac{dt}{(t-x)^3} = \frac{1}{2(1-x)^2}\int_1^n V_x^{x+(1-x)/\sqrt{t}}(g_x)\,dt \leqslant \frac{1}{2(1-x)^2}\sum_{k=1}^{n-1} V_x^{x+(1-x)/\sqrt{k}}(g_x).$$

Therefore

$$|R_n(f,x)| \leqslant \frac{x}{n(1-x)}\left(V_x^1(g_x) + \sum_{k=1}^{n-1} V_x^{x+(1-x)/\sqrt{k}}(g_x)\right) \leqslant \frac{2}{nx(1-x)}\sum_{k=1}^{n} V_x^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.14}$$

From (3.10), (3.11), (3.12), (3.13) and (3.14), it follows that

$$|B_n(g_x, x)| \leqslant \frac{2}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x)$$

$$\leqslant \frac{3}{x(1-x)n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \qquad (3.15)$$

Our theorem now follows from (2.1), (3.9) and (3.15).

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